

TQFT computations and experiments.

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Our recent computer program TQFT allows the actual computation of the projective representation of the mapping class group of a surface on its Verlinde modules. The program is build upon the fusion formulae of Roberts, Masbaum and Vogel [L],[M-V],[R]. The author is greatly indebted to Gregor Masbaum for his explanations during the writing of the program. The program uses the powerful calculator Pari-gp.

The actual program is mainly written for surfaces $\Sigma_{g,1}$ of genus $g \geq 1$ with one boundary component. We think of the surface $\Sigma_{g,1}$ as the boundary of a thickening H_g in \mathbf{R}^3 of the trivalent graph Γ_g of Fig. 1. The graph Γ_g has $3g - 1$ edges. The Verlinde module $V_{k,i}^g$ appears as a vector space over the field $C_{2k+4} = \mathbf{Q}(A)/(\Phi_{2k+4}(A))$, where Φ_{2k+4} is the cyclotomic polynomial, whose roots are the primitive $(2k+4)$ th roots of unity. The space $V_{k,i}^g$ can be viewed as the C_{2k+4} span of the set of admissible (k,i) -colorings of the edges of the graph Γ_g by integers. An admissible (k,i) -coloring of Γ_g is by definition an edge coloring $c : \text{Edge}(\Gamma_g) \rightarrow \mathbf{Z}$ satisfying the following conditions:

1. For each edge e the integer $c(e)$ is even and satisfies $0 \leq c(e) \leq k$;
2. For each node of Γ_g with adjacent edges e_1, e_2, e_3 the colors $c(e_1), c(e_2), c(e_3)$ satisfy the triangular inequalities and the inequality $c(e_1) + c(e_2) + c(e_3) \leq 2k$;
3. The color of the outgoing edge is i .

We describe the action of the mapping class group $\text{Mod}_{g,1}$ on $V_{k,i}^g$ in terms of the actions of the following generators of $\text{Mod}_{g,1}$. Let A_e be (up to isotopy) the simple loop on $\Sigma_{g,1}$ bounding in H_g an embedded disk having one transversal intersection with edge e . Let B_r be the simple loop surrounding the handle r . The group $\text{Mod}_{g,1}$ is generated by the Dehn twists about the A_e and B_r .

We explain some basic possibilities of the program. See the 00README of <http://www.geometrie.ch/TQFT> and also have a look at the explanations

in the comments of the pari-script tqft.gp. The program TQFT can only be called from a Pari session.

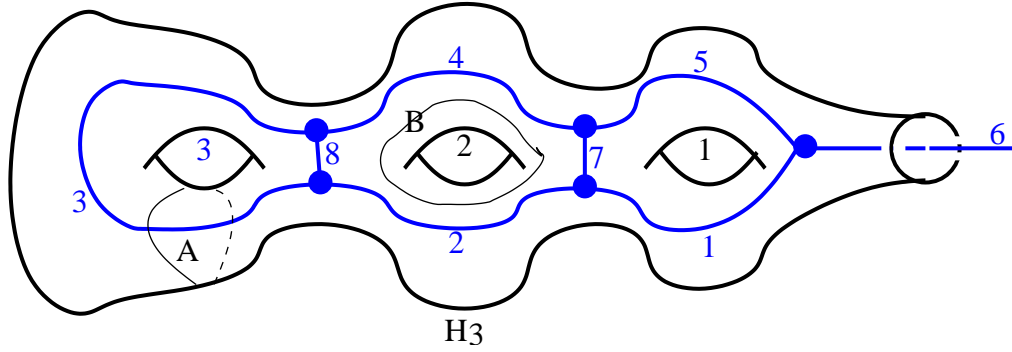


Fig. 1. Handle body of genus 3. Trivalent graph with input edge.

Now start a pari session Pari-gp with the command “gp” and read the file tqft.gp into the pari session. The command

init_so(k)

initializes the global variable A and the field $C_{2k+4} = \mathbf{Q}(A)/(\Phi_{2k+4}(A))$. The command

init_boom_so([0, 1, ..., g - 1], i)

initializes the graph Γ_g and computes the list of its admissible (k, i) -colorings. The commands

twA(e)
twB(r)

compute matrices with coefficients in the field C_{2k+4} of the action of the right Dehn twist about A_e or B_r in the space $V_{k,i}$ with as basis the list of admissible colorings.

Our computations lead to the following results:

Case genus $g = 1$.

In this case the group $\text{Mod}_{g,1}$ is identified via its action on $H_1(\Sigma, \mathbf{Z})$ with $\text{SL}(2, \mathbf{Z})$. The space $V_{k,i}^1$ is generated by the admissible colorings (j, i) of the following graph

The admissibility conditions restrict to:

$$i \leq 2j, i + 2j \leq 2k, j \text{ even}$$

hence we have the colorings

$$(j, i), j = i/2, i/2 + 2, \dots, (k - 1 - i)/2$$

if i and $k - 1$ are divisible by 4,

$$(j, i), j = i/2 + 1, i/2 + 3, \dots, (k - 1 - i)/2$$

, if i nor $k - 1$ are divisible by 4,

$$(j, i), j = i/2, i/2 + 2, \dots, (k - 3 - i)/2$$

if i divisible by 4 but not $k - 1$ and finally

$$(j, i), j = i/2 + 1, i/2 + 3, \dots, (k - 1 - i)/2$$

if k divisible by 4 but not i . So for $k = 2h + 1 \geq 3$ the dimension of the vector space $V_{2h+1,2}^1$ is h .

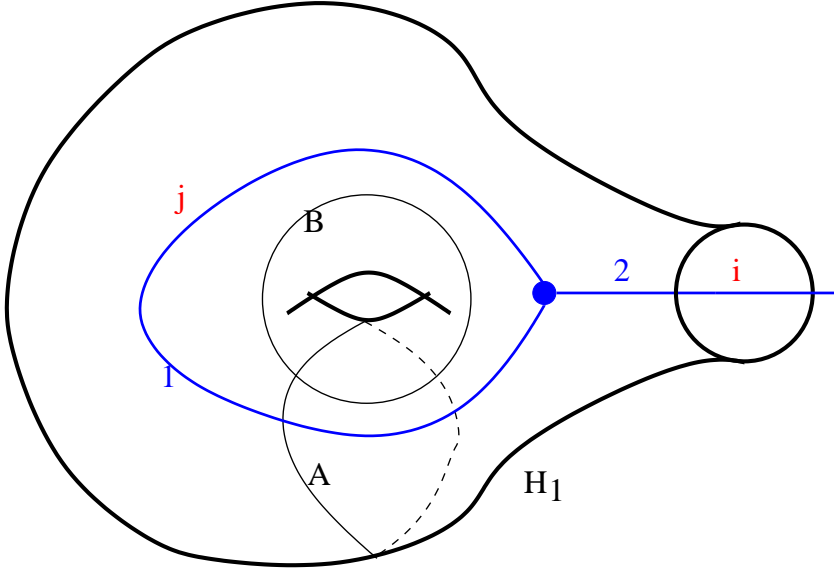


Fig. 2. Handle body of genus 1. Loops A and B .

Let D_1, D_2 be the right Dehn twists about the curves A_1, B_1 of the Fig. 2. The actions in homology $D_{1,*}$ and $D_{2,*}$ for a suitable orientation of the the torus are given by the matrices $D_{1,*} = [1, 1; 0, 1]$ and $D_{2,*} = [1, 0; -1, 1]$. The actions of $V_{5,2}^1$ are obtained by doing first the commands:

init_so(5); init_boom_so([0], 2);

followed by

twA(1)

for $D_{1;5,2}$ and by

twB(1)

for $D_{2;5,2}$. One gets the matrices (remember that the coefficients are in C_{14}):

$$D_{1;5,2} = \begin{pmatrix} A^5 - A^4 + A^3 - A^2 + A - 1 & 0 \\ 0 & A^4 \end{pmatrix}$$

$D_{1;5,2} =$

$$\begin{pmatrix} \frac{3}{7}A^5 + \frac{1}{7}A^4 + \frac{2}{7}A^3 - \frac{5}{7}A^2 + \frac{1}{7}A - \frac{4}{7} & -\frac{4}{7}A^5 + \frac{8}{7}A^4 - \frac{5}{7}A^3 + \frac{2}{7}A^2 - \frac{6}{7}A + \frac{3}{7} \\ -\frac{3}{7}A^5 + \frac{6}{7}A^4 - \frac{2}{7}A^3 + \frac{5}{7}A^2 - \frac{1}{7}A + \frac{4}{7} & \frac{4}{7}A^5 - \frac{1}{7}A^4 + \frac{5}{7}A^3 - \frac{2}{7}A^2 + \frac{6}{7}A - \frac{3}{7} \end{pmatrix}$$

We now change the level to $k = 7$ and work with the representation $\rho_7 : \text{SL}(2, \mathbf{Z}) \rightarrow \text{PGL}(V_{7,2}^1) = \text{PGL}(3, C_{18})$. This requires the commands:

init_so(7); init_boom_so([0], 2);

The image of the right Dehn twist about $A(1)$ is represented by the following matrix with coefficients in $C_{18} = \mathbf{Q}(A)/(A^6 - A^3 + 1)$

$D_{1;7,2} =$

$$\begin{pmatrix} -A & 0 & 0 \\ 0 & -A^3 & 0 \\ 0 & 0 & A^3 - 1 \end{pmatrix}$$

It is interesting to observe that the matrices $D_{1;7,2}$ and its inverse $D_{1;7,2}^{-1}$ do not represent conjugate elements in the group $\text{PGL}(3, C_{18})$, since the class function $m \in \text{PGL}(3, C_{18}) \mapsto \text{trace}(m)^3 / \det(m) \in C_{18}$ takes on $D_{1;7,2}$ the value

$$\text{Mod}(2A^5 - A^2, A^6 - A^3 + 1)$$

and on $D_{1;7,2}^{-1}$ the value

$$\text{Mod}(-A^4 - A, A^6 - A^3 + 1).$$

It follows, as noticed by Vladimir Turaev, that the representation

$$\rho_7 : \text{SL}(2, \mathbf{Z}) \rightarrow \text{PGL}(V_{7,2}^1)$$

does not extend to a representation on $\text{GL}(2, \mathbf{Z})$ since $[1, 1; 0, 1]$ and $[1, -1; 0, 1]$ are in $\text{GL}(2, \mathbf{Z})$ conjugate.

Our next observation is that the images of the matrices $a = [7, 3; 2, 1]$ and $b = [7, 1; 6, 1]$ under $\rho_5 : \text{SL}(2, \mathbf{Z}) \rightarrow \text{PGL}(V_{5,2}^1) = \text{PGL}(2, \mathbf{Q}(A)/(\Phi_{14}(A)))$ are not conjugate in $\text{PGL}(2, \mathbf{Q}(A)/(\Phi_{14}(A)))$. This is worth noticing since the matrices a, b are conjugate in $\text{SL}(2, \mathbf{Q})$.

From this observation we speculate about a positive answer to the following question: given two elements $a, b \in \text{SL}(2, \mathbf{Z})$ that are not conjugate in $\text{SL}(2, \mathbf{Z})$ does there exist $k \geq 3, k \text{ odd}$, such that the images $\rho_k(a), \rho_k(b)$ are not conjugate in $\text{PGL}(2, \mathbf{Q}(A)/(\Phi_{2k+4}(A)))$?

We wish to compute for $a = [7, 3; 2, 1]$ and $b = [7, 1; 6, 1]$ in level $k = 5$ and with input color $i = 2$, so initialize back to $k = 5, i = 2$ with:

`init_so(7); init_boom_so([0], 2);`

The following commands create the matrices a, b :

`a = Mat([7, 3; 2, 1]); b = Mat([7, 1; 6, 1]);`

With the commands:

`wa = slw(a); wb = slw(b);`

we write a, b as products of the matrices $D_{1;*} = [1, 1; 0, 1]$ and $D_{2;*} = [1, 0; 1, -1]$.

$$wa = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right)$$

$$wb = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right)$$

The corresponding products of the matrices $D_{1;5,2}$ and $D_{2;5,2}$ computes the actions Va and Vb of a and b on $V_{1,2}^1$. One gets these matrices directly with the following commands:

$$Va = \text{eval_sl}(a, 5, 2); Vb = \text{eval_sl}(b, 5, 2);$$

$$Va = \begin{pmatrix} \frac{8}{7}A^5 - \frac{2}{7}A^4 + \frac{3}{7}A^3 - \frac{4}{7}A^2 + \frac{5}{7}A - \frac{6}{7} & \frac{1}{7}A^5 + \frac{5}{7}A^4 - \frac{4}{7}A^3 + \frac{3}{7}A^2 - \frac{2}{7}A + \frac{1}{7} \\ \frac{6}{7}A^5 + \frac{2}{7}A^4 + \frac{4}{7}A^3 - \frac{3}{7}A^2 + \frac{2}{7}A - \frac{8}{7} & \frac{6}{7}A^5 + \frac{2}{7}A^4 + \frac{4}{7}A^3 - \frac{3}{7}A^2 + \frac{2}{7}A - \frac{1}{7} \end{pmatrix}$$

$$Vb = \begin{pmatrix} \frac{-4}{7}A^5 + \frac{1}{7}A^4 - \frac{5}{7}A^3 + \frac{2}{7}A^2 + \frac{1}{7}A + \frac{3}{7} & \frac{3}{7}A^5 + \frac{1}{7}A^4 - \frac{5}{7}A^3 + \frac{2}{7}A^2 + \frac{1}{7}A + \frac{3}{7} \\ \frac{4}{7}A^5 - \frac{1}{7}A^4 + \frac{5}{7}A^3 - \frac{2}{7}A^2 - \frac{1}{7}A - \frac{3}{7} & \frac{4}{7}A^5 - \frac{1}{7}A^4 - \frac{5}{7}A^3 - \frac{2}{7}A^2 - \frac{1}{7}A - \frac{3}{7} \end{pmatrix}$$

Our claim is that Va, Vb are not conjugate in $\text{PGL}(2, C_{14})$: The quantities

$$\frac{\text{trace}(Va)^2}{\det(Va)} \in C_{14}, \quad \frac{\text{trace}(Vb)^2}{\det(Vb)} \in C_{14}$$

are conjugacy invariants in the group $\text{PGL}(2, C_{14})$ and we deduce our the claim from

$$\frac{\text{trace}(Va)^2}{\det(Va)} = \text{Mod}(-3A^5 + 2A^4 - 2A^3 + 3A^2 + 5, \Phi_{14}(A)),$$

$$\frac{\text{trace}(Vb)^2}{\det(Vb)} = \text{Mod}(1, \Phi_{14}(A)).$$

The representations $\rho_k : \text{SL}(2, \mathbf{Z}) \rightarrow \text{PGL}(2, \mathbf{Q}(A)/(\Phi_{2k+4}(A)))$ can be lifted to $\text{GL}_1(2, \mathbf{Q}(A)/(\Phi_{2k+4}(A))/\mu_{2k+4})$, hence the maximal absolut value of $\sigma(\text{trace}(\rho_k(a)))$, σ running over all the embeddings of the field $\mathbf{Q}(A)/(\Phi_{2k+4}(A))$ into \mathbf{C} , is an invariant $\|a\|_k$ for a . Here we have denoted by GL_1 the subgroup of elements in GL having a root of unity as determinant.

The maximal absolut value of $\sigma(\text{trace}(\rho_5(a)))$ and of $\sigma(\text{trace}(\rho_5(b)))$ are computed with the commands:

$$\text{normk}(\text{trace}(Va)); \text{normk}(\text{trace}(Vb));$$

We get

$$\|a\|_5 = 2.8019377358048382524722046390148901023$$

$$\|b\|_5 = 1,$$

showing once more that Va, Vb are not conjugate in $\text{PGL}(2, C_{14})$.

Our second experiment concerns the growth of the TQFT action of the matrix $a = [2, 1; 1, 1]$. We have observed that the inequalities $\|a\|_k < \text{trace}(a) = 3, k = 3, 7, 9, \dots$ hold, and that 3 is the supremum of $\{\|a\|_k \mid k \text{ odd}\}$. Here the output of the following command line:

$$(15 : 12) \text{ gp } > \text{for}(i = 1, 16, k = 2 * i + 1; \\ \text{print}(\text{normk}(\text{trace}(\text{eval_sl}([2, 1; 1, 1], k, 2))));$$

1

$$2.2469796037174670610500097680084796213$$

2.8793852415718167681082185546494629398
 2.9189859472289947797807361141326553981
 2.7709120513064197918007510440301977572
 1
 2.8649444588087116091462317836431267725
 2.9727226068054447472050183896381342215
 2.9776616524502570901394857658680172261
 2.9258345746955985900304471947464775986
 1
 2.9460897411596476776657703455693918400
 2.9882759143087192179106054317591031337
 2.9897386467837902926427066197674389859
 2.9638573945254134007973488852494919219
 1
time = 23mn, 6, 060 ms.

Since $V_{5,2}$ has dimension 2, we may deduce from the second line of output $\|a\|_5 = 2.246\dots$, that the action of $a = [2, 1; 1, 1]$ on $V_{5,2}^1$ is a very simple and explicit example of an element of infinite order in TQFT, see [F],[Gi],[M]. Indeed, the product of the two eigenvalues λ_1, λ_2 of $\rho_5(a)$ is a 14th root of unity and for some embedding σ of the field that contains λ_1, λ_2 we have $|\sigma(\lambda_1) + \sigma(\lambda_2)| > 2$; it follows $\max(|\sigma(\lambda_1)|, |\sigma(\lambda_2)|) > 1$. It follows from the corollary of Theorem 2 [Gi] of P. Gilmer that the action of a on $V_{k,0}^1$ any k are periodic.

Case $g > 1$.

The two slalom knots K_1 and K_2 , see [AC1,AC2], of the planar rooted trees $[0, 1, 1, 1, 2]$ and $[0, 1, 1, 1, 3]$ in the table of KNOTSCAPE [H-T] are the knots $13n1320$ and $13n1291$ respectively. According to the author's experience, it is difficult to separate this pair of mutant knots by invariants. They have for instance, equal Kauffman polynomial and HOMFLY polynomial. Moreover, the Khovanov homologies coincide, as one can verify using the program KhoHo of Alexander Shumakovitch, see <http://www.geometrie.ch/KhoHo>. With SNAPPEA [W] we were able to show that these knots have non isomorphic rigid symmetry groups. With SNAP [G] we could not find a distinction based on arithmetic properties of the hyperbolic structures on the complements. The knots are fibered with fibers of genus 5 and the monodromy diffeomorphisms can be written explicitly in terms of the underlying planar rooted trees as products of Dehn twist.

The program TQFT was written and especially designed in order to compute the action of monodromies of slalom knots in TQFT. The following command line computes the TQFT actions for level $k = 3$ and input color $i = 2$ on $V_{3,2}^5$:

$$c_1 = \text{coxeter}([0, 1, 1, 1, 2], 3, 2); c_2 = \text{coxeter}([0, 1, 1, 1, 3], 3, 2);$$

The computation is finished after: *time* = 5mn, 930 ms. We do not ask for an output on the screen, since c_1, c_2 are square matrices of size 275 representing elements in $\text{PGL}(275, C_{10})$. First we compute the traces and get:

$$\text{trace}(c_1) = \text{Mod}(7A^3 + A^2 + 5A - 4, A^4 - A^3 + A^2 - A + 1),$$

$$\text{trace}(c_2) = \text{Mod}(7A^3 + A^2 + 5A - 4, A^4 - A^3 + A^2 - A + 1).$$

So we have equal non vanishing traces. We compute traces of iterates:

$$\text{trace}(c_1^2) = \text{Mod}(-13A^3 + 18A^2 + 4A + 25, A^4 - A^3 + A^2 - A + 1),$$

$$\text{trace}(c_2^2) = \text{Mod}(-13A^3 + 18A^2 + 4A + 25, A^4 - A^3 + A^2 - A + 1).$$

Again equal, so, we continue with traces of third powers:

$$\text{trace}(c_1^3) = \text{Mod}(-47A^3 - 56A^2 - 65A - 2, A^4 - A^3 + A^2 - A + 1),$$

$$\text{trace}(c_2^3) = \text{Mod}(-62A^3 - 47A^2 - 68A - 6, A^4 - A^3 + A^2 - A + 1).$$

Third power traces are different, so since the traces of first powers are equal and non-vanishing, we conclude that the TQFT-actions of the monodromies of the knots K_1, K_2 are not conjugate in $\text{PGL}(275, C_{10})$.

We wish to compute for the two knots K_1, K_2 also with input color $i = 0$, so we use the commands:

$$c_1 = \text{coxeter}([0, 1, 1, 1, 2], 3, 0); c_2 = \text{coxeter}([0, 1, 1, 1, 3], 3, 0);$$

which was done after *time* = 2mn, 18, 840 ms. This corresponds to a study of the monodromy as a diffeomorphism of the closed surface of genus 5. We now get matrices of smaller size 175 and as before, the separation of the knots with traces of third powers:

$$\text{trace}(c_1) = \text{Mod}(5A^3 + 3A - 3, A^4 - A^3 + A^2 - A + 1),$$

$$\text{trace}(c_2) = \text{Mod}(5A^3 + 3A - 3, A^4 - A^3 + A^2 - A + 1),$$

$$\text{trace}(c_1^2) = \text{Mod}(-8A^3 + 11A^2 + 3A + 15, A^4 - A^3 + A^2 - A + 1),$$

$$\text{trace}(c_2^2) = \text{Mod}(-8A^3 + 11A^2 + 3A + 15, A^4 - A^3 + A^2 - A + 1),$$

$$\text{trace}(c_1^3) = \text{Mod}(-20A^3 - 37A^2 - 35A - 25, A^4 - A^3 + A^2 - A + 1),$$

$$\text{trace}(c_2^3) = \text{Mod}(-29A^3 - 31A^2 - 38A - 13, A^4 - A^3 + A^2 - A + 1).$$

The two slalom knots of the planar rooted trees $[0, 1, 1, 3, 3, 4]$ and $[0, 1, 1, 3, 3, 5]$ are the knots 15n30444 and 15n30419 respectively. For this pair we have again that the traces of the first and second powers of the actions of the monodromies in level $k = 3$ with input color $i = 0$ on $V_{3,0}^6$ coincide, and that the traces of the third powers distinguish the knots of this pair. For this

calculation the size of the matrices grew up to 675.

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